LINEAR DIVISIBILITY SEQUENCES*

BY

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I. Introduction

1. A sequence of rational integers

$$(u): \qquad u_0, u_1, \cdots, u_n, \cdots$$

is called a *divisibility sequence* if u_n divides u_m whenever n divides m. (u) is *linear*† if it satisfies a linear difference equation with integral coefficients and *normal* if $u_0=0$, $u_1=1$. Marshall Hall has shown that a linear divisibility sequence is usually normal [2]. If

$$f(x) = x^k - c_1x^{k-1} - \cdots - c_k, \qquad c_1, \cdots, c_k \text{ integers}, \tag{1.1}$$

is the polynomial associated with the difference equation of lowest order which (u) satisfies, (u) is said to be of *order* k and to *belong to* its *characteristic polynomial* $f(x)$.

An integer dividing every term of (u) beyond a certain point is called a *null divisor* of (u) [3]. If (u) has no null divisors save ± 1, it is said to be *primary*.

If u_s is any fixed non-vanishing term of (u), the sequence

$$u_0/u_s, u_s/u_s, u_{2s}/u_s, \cdots, u_{ns}/u_s, \cdots$$

is called a *subsequence* of (u). The various subsequences of (u) are themselves normal linear divisibility sequences of order $\leqq k$.

2. The object of this paper is to prove the following results:

Let the characteristic polynomial of the linear divisibility sequence (u) have no repeated roots, and let its coefficients be relatively prime. Then:

I. *If (u) is primary and if q is any large prime number,*

$$u_q^\sigma \equiv 1 \pmod{q}, \tag{2.1}$$

where σ is the least common multiple of $1, 2, 3, \cdots, k$.

II. *If (u) is not primary it always contains an infinity of subsequences which are primary. Furthermore the characteristic polynomials of such subsequences satisfy the hypotheses imposed above upon the polynomial* (1.1).

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† T. A. Pierce appears to have been the first to discuss sequences of order greater than two [1]. (Numbers in square brackets refer to the bibliography at the end of the paper.)

III. *There exists a rational number*

$$B = B(u) = B(u_0, u_1, \cdots, u_{k-1}; c_1, \cdots, c_k) = \frac{P}{Q}, \qquad (P, Q) = 1$$

such that

(i) *if p is a prime number dividing neither the numerator P nor the denominator Q of B, then the rank of apparition* of p in the sequence (u) is the restricted period* of (u) modulo p;*

(ii) *the prime factors of the denominator of B all divide the discriminant of the polynomial to which (u) belongs;*

(iii) *the numerator of B can never vanish if the galois group of $f(x)$ is alternating or symmetric.*†

II. Proof of first result

3. Given any modulus m, the least period of (u) modulo m is called its characteristic number and the number of non-periodic terms in (u) modulo m its numeric. The reader will be assumed to be familiar with my previous paper in these Transactions [4] (referred to hereafter as T) devoted to the determination of these numbers.

Henceforth let (u) be a normal linear divisibility sequence of order k, and let D denote the discriminant of its characteristic polynomial. We assume:

$$D \neq 0. \tag{3.1}$$

Lemma 3.1 [4]. *If*‡ $(q, D) = 1$, *q a prime, and if σ is the least common multiple of* $2, 3, \cdots, k$, *then (u) admits the period $q^\sigma - 1$ modulo q.*

Theorem 3.1. *If (u) is a linear divisibility sequence of order k and q a prime such that $u_q \equiv 0 \pmod{q}$, then either q divides D or q divides c_k.*

Assume that‡ $q \mid u_q$, q a prime. The assumption $(q, c_k) = (q, D) = 1$ then yields a contradiction. For if $(q, c_k) = 1$, (u) is purely periodic modulo q [5]. And if $(q, D) = 1$, (u) admits the period $q^\sigma - 1$ modulo q. Determine positive integers x and y such that $xq = y\,(q^\sigma - 1) + 1$. Then $u_{xq} \equiv u_1 \equiv 1 \pmod{q}$. But $q \mid u_q$ and $u_q \mid u_{xq}$.

The following lemma is a direct consequence of Theorem 3.1.

* The rank of apparition of p is the index ρ of the first term of (u) excluding u_0 which divides: $u_\tau \equiv 0 \pmod{p}$; $u_n \not\equiv 0 \pmod{p}$, $0 < n < \rho$. The restricted period [5] of (u) modulo is the least positive integer τ such that $u_{n+\tau} \equiv c u_n \pmod{p}$, $n = 0, 1, 2, \cdots$, c an integer. ρ always divides τ [2].

† It is unknown whether divisibility sequences exist whose characteristic polynomial is restricted as in (iii). No such sequences exist when $k = 3$ [2].

‡ If $a, b, c, \cdots$ are rational integers, we write as usual $(a, b, c, \cdots)$ for the greatest common divisor of $a, b, c, \cdots$, and $a \mid b$ for a divides b.

LEMMA 3.2. *There exists a rational integer q_0 such that*

$$u_q \not\equiv 0 \pmod{q}, \qquad q \text{ a prime} \geqq q_0. \tag{3.2}$$

LEMMA 3.3 [4]. *For any prime p, $p^k(p^\sigma - 1)$ is a period of (u) modulo p.*

LEMMA 3.4 [4]. *For any prime p, the numeric of (u) modulo p is less than or equal to k.*

THEOREM 3.2. *If p is a prime dividing a term u_q of the divisibility sequence (u) with a sufficiently large prime index q, then either*

$$p^\sigma \equiv 1 \pmod{q} \tag{3.3}$$

or else (u) is a null sequence modulo p.

Choose a prime $q > k$ and q_0 of (3.2), and assume that $u_q \equiv 0 \pmod{p}$, p a prime. By (3.2), $p \neq q$. Hence if $(p^\sigma - 1, q) = 1$, for each positive integer r there exist positive integers x, y, z such that

$$xq + yp^k(p^\sigma - 1) = r + zp^k(p^\sigma - 1). \tag{3.4}$$

By Lemma 3.3, $p^k(p^\sigma - 1)$ is a period of (u) modulo p. Therefore if $r > k$, (3.4) and Lemma 3.4 give $u_{xq} \equiv u_r \pmod{p}$. Since $p \mid u_q$ and $u_q \mid u_{xq}$, $u_r \equiv 0 \pmod{p}$ so that (u) is a null sequence modulo p.

THEOREM 3.3. *If the linear divisibility sequence (u) is primary, and if k is its order and σ the least common multiple of the numbers $2, 3, \cdots, k$, then for all sufficiently large prime indices q we have*

$$u_q^\sigma \equiv 1 \pmod{q}. \tag{2.1}$$

Choose the prime $q > k$ and q_0 of (3.2), and let the factorization of u_q be $u_q = p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t}$. Since (u) is assumed primary none of the primes p_i are null divisors. Therefore Theorem 3.2, $p_i^\sigma \equiv 1 \pmod{q}$, so that

$$p_i^{\sigma s_i} \equiv 1 \pmod{q}, \qquad (i = 1, 2, \cdots, t).$$

On multiplying these t congruences together, we obtain (2.1), and our first result is proved.

III. PROOF OF SECOND RESULT

4. We assume that (u) is a normal linear divisibility sequence for which

$$(c_1, c_2, \cdots, c_k) = 1. \tag{4.1}$$

A *proper* null divisor of a linear sequence is one which divides neither its initial terms nor the coefficients of its recursion. Any other null divisor is called *trivial*. (u) obviously has no trivial null divisors.

THEOREM 4.1. *No subsequence of* (u) *has trivial null divisors.*

LEMMA 4.1 (Schatanovskis Principle) [6, 7, 8]. *If* $\Phi(x_1, x_2, \cdots, x_k)$ *is an integral symmetric function of the arguments* $x_1, \cdots, x_k$ *with integral coefficients, and if for a natural number* m

$$f(x) \equiv (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k) \equiv (x-\gamma_1)(x-\gamma_2)\cdots(x-\gamma_k) \pmod{m},$$

where $f(x)$ *is a polynomial with integral coefficients, then*

$$\Phi(\alpha_1, \alpha_2, \cdots, \alpha_k) \equiv \Phi(\gamma_1, \gamma_2, \cdots, \gamma_k) \pmod{m}.$$

LEMMA 4.2. *Let*

$$f^{(s)}(x) = x^k - d_1x^{k-1} - \cdots - d_k$$

be the polynomial whose roots are the sth powers of the roots of $f(x)$, *and* p *a prime number. Then if* t *is any positive integer* $\leqq k$, (A) $p \mid (c_k, c_{k-1}, \cdots, c_{k-t+1})$ *when and only when* (B) $p \mid (d_k, d_{k-1}, \cdots, d_{k-t+1})$.

Assume that (A) holds. Then

$$f(x) \equiv g(x) = x^{k-t}(x^t - c_1x^{t-1} - \cdots - c_{k-t}) \pmod{p}.$$

Let the k roots of $g(x)=0$ be $\gamma_1, \gamma_2, \cdots, \gamma_t; \gamma_{t+1}=\gamma_{t+2}= \cdots =\gamma_k=0$. If the roots of $f(x)=0$ are $\alpha_1, \alpha_2, \cdots, \alpha_k$, then $d_i=\Phi(\alpha_1, \alpha_2, \cdots, \alpha_k)$, where Φ is a symmetric polynomial in its arguments with rational integral coefficients. Hence by the preceding lemma

$$d_i \equiv \Phi(\gamma_1, \gamma_2, \cdots, \gamma_k) \pmod{p}.$$

But if $g^{(s)}(x)=x^k-e_1x^{k-1}- \cdots -e_k$ is the equation whose roots are the sth powers of the roots of $g(x)=0$, then

$$e_i = \Phi(\gamma_1, \gamma_2, \cdots, \gamma_k) = \Sigma\gamma_1^s\gamma_2^s \cdots \gamma_i^s = 0 \text{ if } i > k-t.$$

Hence $d_i \equiv 0 \pmod{p}$ if $i>k-t$, so that (B) follows.

To prove the converse, it suffices to show that (A) and $c_{k-t}\not\equiv 0 \pmod{p}$ imply that $d_{k-t}\not\equiv 0 \pmod{p}$. But by what precedes,

$$d_{k-t} \equiv \Sigma(\gamma_1\gamma_2\cdots\gamma_t)^s \equiv (\gamma_1\gamma_2\cdots\gamma_t)^s \equiv c_{k-t}^s \pmod{p}.$$

Proof of Theorem 4.1. With the notation of Lemma 4.2, any subsequence (v): $v_n=u_{ns}/u_s$ of (u) is normal, so that the only possible trivial null divisors of (v) are common divisors of $d_1, d_2, \cdots, d_k$. On taking $t=k$ in Lemma 4.2, we see that if $(c_1, c_2, \cdots, c_k)=1$ then $(d_1, d_2, \cdots, d_k)=1$.

5. We begin our discussion of the proper null divisors of (u) by restating some properties of linear sequences used in T. Let

$$f_0(x) = 0, \quad f_r(x) = x^r - c_1x^{r-1} - \cdots - c_r, \qquad (r = 1, 2, \cdots, k).$$

The polynomial

$$u(x) = u_0 f_{k-1}(x) + u_1 f_{k-2}(x) + \cdots + u_{k-1} f_0(x) \tag{5.1}$$

is called the *generator* of the sequence (u).* If furthermore

$$\Delta(u) = \begin{vmatrix} u_0, & u_1, \cdots, & u_{k-1} \\ u_1, & u_2, \cdots, & u_k \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ u_{k-1}, & u_k, \cdots, & u_{2k-2} \end{vmatrix}, \tag{5.2}$$

then

$$\Delta(u) = (-1)^{k(k-1)/2} \operatorname{Res} \{u(x), f(x)\} = \beta_1\beta_2 \cdots \beta_k D, \tag{5.3}$$

where $u_n = \beta_1\alpha_1^n + \cdots + \beta_k\alpha_k^n$ and $\alpha_1, \cdots, \alpha_k$ are the roots of $f(x)$. Since (u) is of order k and $D \neq 0$, $\Delta(u) \neq 0$.

Consider next the $k+1$ greatest common divisors

$$\begin{aligned} e_0 &= (u_0, u_1, u_2, \cdots, u_{k-1}) \\ e_1 &= (c_k, u_1, u_2, \cdots, u_{k-1}) \\ e_2 &= (c_k, c_{k-1}, u_2, \cdots, u_{k-1}) \\ &\cdots\cdots\cdots\cdots \\ e_{k-1} &= (c_k, c_{k-1}, c_{k-2}, \cdots, u_{k-1}) \\ e_k &= (c_k, c_{k-1}, c_{k-2}, \cdots, c_1). \end{aligned}$$

Then

$$e_0 = e_1 = e_k = 1.$$

The following lemma easily follows from formula (5.1) and the results of part IV of T.

LEMMA 5.1. *Necessary and sufficient conditions that a linear sequence of order k be primary are that the $k+1$ greatest common divisors e_i be all equal to unity.*

THEOREM 5.1. *If the prime p is a null divisor of the normal linear divisibility sequence (u), then p divides both $\Delta(u)$ and the discriminant D of the characteristic polynomial $f(x)$ of (u).*

It is easily shown that every such p must divide one or the other of the numbers e_i. Since $e_k = 1$, $p \mid u_{k-1}$. Hence $p \mid u_k$, $p \mid u_{k+1}, \cdots$ by Lemma 3.4.

* We have the identity $u(x)/f(x) = \sum_0^\alpha u_n/x^{n+1}$ for $|x|$ large. See T, p. 606, and [3].

Hence $p|\Delta(u)$ by formula (5.2). Since $e_0=e_1=1$, $p|c_k$ and $p|c_{k-1}$. Hence $x=0$ is a multiple root of the congruence $f(x)\equiv 0 \pmod p$ and $p|D$.

As a corollary, we have

LEMMA 5.2. *A sufficient condition that the divisibility sequence (u) be primary is that D and $\Delta(u)$ be co-prime.*

If p is a prime proper null divisor of (u), the exponent of the highest power of p which is a null divisor of (u) is called the *index* of p in (u) [3].

LEMMA 5.3 [3]. *Let (u) be a linear sequence for which* (4.1) *holds. Then the index of any prime null divisor p is $\leqq r$, where p^r is the highest power of p dividing $\Delta(u)$.*

THEOREM 5.2. *A subsequence of a normal linear divisibility sequence can have no prime null divisor which is not a possible null divisor of (u) itself.*

Every prime null divisor of (u) must divide c_k in (1.1) [5]. Let (v) be any subsequence of (u). By Theorem 4.1, (v) can have only proper null divisors. Hence any prime null divisor of (v) must divide the constant term d_k of the polynomial to which (v) belongs. But obviously d_k divides some power of c_k.

6. Let $f^{(s)}(x)=(x-\alpha_1^s)\cdots(x-\alpha_k^s)$ be the polynomial whose roots are the sth powers of the roots of $f(x)$, and let $D^{(s)}$ be its discriminant. $D^{(s)}/D$ is clearly an integer.

THEOREM 6.1. *The integer s may be chosen in an infinite number of ways so that $D^{(s)}/D$ is prime to D.*

Let p be any prime factor of D, $\mathfrak{F}$ the Galois field of $f(x)$, and $\mathfrak{p}$ a prime ideal factor of p in $\mathfrak{F}$. Then since $D^{1/2}=\prod_{i<j}(\alpha_i-\alpha_j)$, $p|D$ only when $\alpha_i-\alpha_j\equiv 0 \pmod{\mathfrak{p}}$ for some values of the subscripts i and j.

Now

$$\left(\frac{D^{(s)}}{D}\right)^{1/2}=\prod_{i<j}\frac{\alpha_i^s-\alpha_j^s}{\alpha_i-\alpha_j}\quad\text{and}\quad\frac{\alpha_i^s-\alpha_j^s}{\alpha_i-\alpha_j}\equiv s \pmod{[\alpha_i-\alpha_j]}.^*$$

Hence if $\alpha_i-\alpha_j\equiv 0 \pmod{\mathfrak{p}}$, then $\alpha_i^s-\alpha_j^s/(\alpha_i-\alpha_j)\equiv 0 \pmod{\mathfrak{p}}$ if and only if $s\equiv 0 \pmod{\mathfrak{p}}$; that is, if and only if $s\equiv 0 \pmod p$. Choose s prime to D. Then if $D^{(s)}/D$ and D have a common factor, and hence a common prime factor p, we must have for some k and l

$$(6.1)\qquad \alpha_k^s\equiv\alpha_l^s \pmod{\mathfrak{p}},\qquad\qquad (6.11)\qquad \alpha_k\not\equiv\alpha_l \pmod{\mathfrak{p}},$$

where $\mathfrak{p}|p$. If both (6.1) and (6.11) hold, then

* The square bracket denotes a principal ideal.

$$(\alpha_k, \mathfrak{p}) = (\alpha_l, \mathfrak{p}) = (\alpha_k - \alpha_l, \mathfrak{p}) = \mathfrak{o}, \tag{6.2}$$

where $\mathfrak{o}$ as usual is the unit ideal of $\mathfrak{F}$.

Now for each pair of distinct roots α_i, α_j of $f(x)$ for which $(\alpha_i, \mathfrak{p}) = (\alpha_j, \mathfrak{p}) = (\alpha_i - \alpha_j, \mathfrak{p}) = \mathfrak{o}$, let s_{ij} be the least positive integer y such that

$$\alpha_i^y \equiv \alpha_j^y \pmod{\mathfrak{p}}. \tag{6.3}$$

Then s_{ij} divides every other such y, and in particular the number $N(\mathfrak{p}) - 1 = p^t - 1$. Here $t \leqq k!$, the maximum possible degree of $\mathfrak{F}$.

Let m_p be the least common multiple of the numbers $p-1, p^2-1, \cdots, p^{k!}-1$ and if D has in all k distinct prime factors $p_1, p_2, \cdots, p_k$ let M be the least common multiple of $m_{p_1}, m_{p_2}, \cdots, m_{p_k}$. Then if *$s$ is chosen prime to both M and D* (and this choice can be made in an infinity of ways), *$D^{(s)}/D$ is prime to D.*

For if $(s, D) = 1$ and $(D^{(s)}/D, D) \neq 1$, (6.1) holds. Then $s_{kl} | s$. Since $(s, M) = 1$ and $s_{kl} | M$, $s_{kl} = 1$ contradicting (6.11).

7. As in §6, let $p_1, \cdots, p_k$ be the distinct prime factors of D. By Theorems 4.5, 5.1 and Lemma 5.4, these primes are the only possible prime null divisors of (u) and its subsequences. Write

$$\Delta(u) = p_1^{r_1} \cdots p_k^{r_k} q, \qquad (q, D) = 1, \qquad r_i \geqq 0,$$

and let θ_i be the index of p_i in (u), where if p_i is not a null divisor, $\theta_i = 0$. By Lemma 5.3, $0 \leqq \theta_i \leqq r_i$, $(i = 1, 2, \cdots, k)$.

Now if R is the largest of $r_1, r_2, \cdots, r_k$, the numeric of p^{θ_i} is always less than kR^s. Choose $s > kR$ as in Theorem 6.1, and let (v) be the subsequence of (u) with general term $v_n = u_{ns}/u_s$ belonging to the polynomial $f^{(s)}(x)$. As in Theorem 6.1, let the discriminant of $f^{(s)}(x)$ be $D^{(s)}$. Then since $u_{ns} = \beta_1\alpha_1^{ns} + \cdots + \beta_k\alpha_k^{ns}$, we have by formula (5.3),

$$\Delta(v) = \frac{\Delta(u)}{u_s^k} \frac{D^{(s)}}{D}. \tag{7.1}$$

Now $u_s \equiv 0 \pmod{p_i^{\theta_i}}$ and $(p_i, D^{(s)}/D) = 1$. Hence since $\Delta(v)$ is an integer, $\Delta(u) \equiv 0 \pmod{p_i^{k\theta_i}}$. Therefore $r_i \geqq k\theta_i$. If $\Delta(v) = p_1^{r_1'} \cdots p_k^{r_k'} q'$, $(q', D) = 1$, then $r_i' = r_i - k\theta_i$. Therefore

$$r_i' < r_i \text{ if } \theta_i > 0; \; r_i' = r_i \text{ if } \theta_i = 0. \tag{7.2}$$

8. We now prove our second result indirectly. Suppose that the result is false. Then in any infinite set of normal divisibility sequences

$$\mathfrak{S}: \quad (u^{(1)}) = (u), (u^{(2)}), (u^{(3)}), \cdots, (u^{(m)}), (u^{(m+1)}), \cdots,$$

such that each sequence is a subsequence of its immediate predecessor, there must occur an infinity of non-primary sequences. Therefore there must exist a prime p dividing D which is a null divisor of an infinite number of the sequences $(u^{(m)})$. The general term of $(u^{(m+1)})$ is of the form $u_n^{(m+1)} = u_{ns_m}^{(m)}/u_{s_m}^{(m)}$, where the integer s_m specifies the particular subsequence of $(u^{(m)})$ selected. Consider now a set $\mathfrak{S}$ in which each $u^{(m)}$ satisfies the conditions imposed upon s in §6.

The considerations of the preceding section carry over to the relationship between $(u^{(m)})$ and $(u^{(m+1)})$. With an obvious extension of notation, let $\theta^{(m)}$ denote the index of p in $(u^{(m)})$ and p^{r_m} and $p^{r_{m+1}}$ the highest powers of p dividing $\Delta(u^{(m)})$ and $\Delta(u^{(m+1)})$. Then as in (7.2)

$$r_{m+1} < r_m \text{ if } \theta^{(m)} > 0; \; r_{m+1} = r_m \text{ if } \theta^{(m)} = 0. \tag{8.1}$$

By our hypothesis, an infinite number of the $\theta^{(m)}$ are positive. But then (8.1) leads to an absurdity; for obviously $r = r_1 \geqq r_2 \geqq r_3 \geqq \cdots \geqq 0$.

IV. Proof of third result

9. We assume as in the previous proofs that $D \neq 0$. In the Galois field $\mathfrak{F}$ of $f(x)$, a rational prime p which does not divide D remains unramified [9]. Accordingly the decomposition of p into prime ideal factors in $\mathfrak{F}$ is of the form

$$p = \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_l,$$

where the $\mathfrak{p}$ are all distinct.

Let σ_i be the least positive integer n such that

$$\alpha_1^n \equiv \alpha_2^n \equiv \cdots \equiv \alpha_k^n \pmod{\mathfrak{p}_i} \qquad (i = 1, \cdots, l). \tag{9.1}$$

The restricted period τ of (u) modulo p is defined as the least value of n such that

$$u_{n+m} \equiv a u_n \pmod{p} \qquad (m = 0, 1, 2, \cdots),$$

where a is some rational integer [5]. If p is prime to $\Delta(u)$, τ may be equally defined as the least positive integer n such that we have in $\mathfrak{F}$

$$\alpha_1^n \equiv \alpha_2^n \equiv \cdots \equiv \alpha_k^n \pmod{p}.$$

The following lemma therefore follows.

Lemma 9.1. *If p is a prime dividing neither $\Delta(u)$ nor D, then the restricted period τ of (u) modulo p is the least common multiple of the numbers $\sigma_1, \sigma_2, \cdots, \sigma_l$ associated with the congruence* (9.1) *above.*

10. Since $u_n = \beta_1\alpha_1 + \cdots + \beta_k\alpha_k^n$ and the α_i are distinct,

$$\beta_i = u(\alpha_i)/f'(\alpha_i) \neq 0, \qquad (i = 1, \cdots, k). \tag{10.1}$$

Here $u(x)$ is the generator of the sequence (u) and $f'(x)$ the derivative of $f(x)$. Since $D=f'(\alpha_1)f'(\alpha_2)\cdots f'(\alpha_k)$, every prime ideal factor of the denominators of the β_i divides D. Let p be a rational prime. Since $\Delta(u)=\beta_1\beta_2\cdots\beta_k D$ $=u(\alpha_1)u(\alpha_2)\cdots u(\alpha_k)$ we can state

LEMMA 10.1. *If* $(p, D)=(p, \beta_1\beta_2\cdots\beta_k)=1$, *then* $(p, \Delta(u))=1$. *Conversely if* $(p, \Delta(u))=(p, D)=1$, *then* $(p, \beta_1\beta_2\cdots\beta_k)=1$, *$p$ a rational prime.*

Form the k sets of sums of the β taken $1, 2, \cdots, k$ at a time:

$$\beta_1+\beta_2+\cdots+\beta_i, \cdots, \beta_{k-i+1}+\beta_{k-i+2}+\cdots+\beta_k, \tag{10.2}$$

where $(i=1, 2, \cdots, k)$, and each set contains ${}_kC_i$ summands, not necessarily all distinct. Then take the symmetric products over each set

$$\mathrm{B}_i=\prod(\beta_1+\beta_2+\cdots+\beta_i) \qquad (i=1, 2, \cdots, k;\ \mathrm{B}_k=0).$$

Finally let $B=B(u)=\mathrm{B}_1\mathrm{B}_2\cdots\mathrm{B}_{k-1}$. Then B is a rational number of the form P/Q, where P and Q are integers, co-prime if $B\neq 0$ and the only primes dividing Q are divisors of D.

THEOREM 10.1. *If a prime p divides neither numerator nor denominator of B, its rank of apparition is the restricted period of (u) modulo p.*

By Lemma 10.1 any such prime p satisfies the hypothesis of Lemma 9.1. Let $\mathfrak{p}$ be any prime ideal factor of p, and ρ the rank of apparition of p. Since $p\,|\,u_\rho$ implies that $p\,|\,u_{n\rho}$,

$$\beta_1\alpha_1^{n\rho}+\beta_2\alpha_2^{n\rho}+\cdots+\beta_k\alpha_k^{n\rho}\equiv 0 \pmod{\mathfrak{p}} \qquad (n=0, 1, \cdots, k-1). \tag{10.3}$$

Since the β_i are integers modulo $\mathfrak{p}$ and prime to $\mathfrak{p}$, the determinant of this set of congruences is divisible by $\mathfrak{p}$. But this determinant is the difference product of the numbers $\alpha_1^\rho, \alpha_2^\rho, \cdots, \alpha_k^\rho$. Hence these numbers are not all distinct modulo p. *I say that*

$$\alpha_1^\rho\equiv\alpha_2^\rho\equiv\cdots\equiv\alpha_k^\rho \pmod{\mathfrak{p}}. \tag{10.4}$$

Otherwise the numbers can be grouped into two or more sets:

$$\alpha_{i_1}^\rho\equiv\alpha_{i_2}^\rho\equiv\cdots\equiv\alpha_{i_s}^\rho\equiv\zeta_1 \pmod{\mathfrak{p}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_{j_1}^\rho\equiv\alpha_{j_2}^\rho\equiv\cdots\equiv\alpha_{j_t}^\rho\equiv\zeta_m \pmod{\mathfrak{p}}$$

such that the ζ are all distinct modulo p. The congruences (10.3) can then be replaced by

$$\beta_1'\zeta_1^n+\beta_2'\zeta_2^n+\cdots+\beta_m'\zeta_m^n\equiv 0 \pmod{\mathfrak{p}},$$

where the β' occur in the sets (10.2) of sums of β's. The determinant of the first m of these congruences as the difference product of the ζ is prime to $\mathfrak{p}$. Thus $\beta_1' \equiv \beta_2' \equiv \cdots \equiv \beta_m' \equiv 0 \pmod{\mathfrak{p}}$, so that $p \mid B$, contrary to hypothesis.

From (10.4) and the definition of the numbers σ in §9, we see that $\sigma \mid \rho$. Since this argument applies to all of the prime ideal factors of p, the least common multiple of $\sigma_1, \cdots, \sigma_l$ divides ρ. That is, by Lemma 9.1, $\tau \mid \rho$. But ρ always divides $\tau[2]$. Hence $\rho = \tau$.

LEMMA 10.1. *If the number $B = B(u)$ is not zero, the rank of apparition of all save a finite number of primes in (u) is their restricted period.*

11. We now prove

THEOREM 11.1. *A sufficient condition that the number B be not zero is that the group of the characteristic polynomial of (u) be either alternating or symmetric.*

If B vanishes, one of the numbers of the set (10.2) vanishes. With a proper choice of notation we may assume that*

$$\beta_1 + \beta_2 + \cdots + \beta_i = 0, \qquad (k/2 \leqq i \leqq k). \tag{11.1}$$

We may also assume that $k > 4$, as the cases $k = 2, 3, 4$ may be easily discussed directly (see next theorem). Hence $i \geqq 3$.

If we represent the Galois group $\mathfrak{G}$ of $f(x)$ as a permutation group upon the k roots $\alpha_1, \cdots, \alpha_k$, then formula (10.1) shows that any permutation of the α induces the corresponding permutation upon the β. If $\mathfrak{G}$ is alternating or symmetric, it contains the permutation $S = (\alpha_1\alpha_{i+1})(\alpha_2\alpha_3)$. On applying S to (11.1), we obtain $\beta_{i+1} + \beta_2 + \beta_3 + \cdots + \beta_i = 0$. Hence $\beta_1 = \beta_{i+1}$. Similarly, $\beta_2 = \beta_{i+1}, \cdots, \beta_i = \beta_{i+1}$. Hence $\beta_{i+1} = 0$ contrary to (10.1).

The following result is proved by similar reasoning.

THEOREM 11.2. *For low orders of (u), sufficient conditions that $B(u) \neq 0$ are as follows:*

Order of (u)	*Condition of Galois group or characteristic polynomial*
2, 3	*none*
4	*order of group divisible by* 3
5	*$f(x)$ irreducible, or product of an irreducible quartic and linear factor*
6, 7	*group transitive and primitive.*

* It will be recalled that $\beta_1 + \beta_2 + \cdots + \beta_k = u_0 = 0$.

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